ISRAEL JOURNAL OF MATHEMATICS, Vol. 36, Nos. 3-4, 1980

ON A SEMILINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATION

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ABSTRACT
The Volterra integrodifferential equation

$$u_t(t,x)+\int_0^t a(t-s)(-\Delta u(s,x)+f(x,u(s,x)))ds = h(t,x),$$

$$t>0, \quad x\in\Omega\subset R^N,$$

together with boundary and initial conditions is considered. The existence of global solutions (in time) is established under weak assumptions on f. An application in heat flow is also indicated.

1. Introduction

This paper is concerned with the existence of solutions of the boundary initial value problem

$$(1.1)\quad u_t(t,x)+\int_0^t a(t-s)(-\Delta u(s,x)+f(x,u(s,x)))ds = h(t,x), \quad t>0, \quad x\in\Omega,$$

$$(1.2)\qquad u(t,x)=0, \qquad t\geqq 0, \quad x\in\Gamma,$$

$$(1.3)\qquad u(0,x)=u_0(x), \qquad x\in\Omega.$$

Here Ω is a bounded open subset of R^N with smooth boundary Γ, $a:[0,\infty)\to R$, $h:[0,\infty)\times\Omega\to R$, $u_0:\Omega\to R$ are given functions, Δ denotes the Laplacian, while $f:\Omega\times R\to R$ is a strongly nonlinear perturbing term in the sense we do not impose an over-all growth condition on the size of $f(x,u)$ as a function of u. The problem serves as a very special model for three-dimensional heat flow in materials with memory. This aspect is discussed in Section 5. It also occurs in viscoelasticity (see [11]).

In case f is independent of x, continuous and monotone nondecreasing in u,

Received April 22, 1979 and in revised form October 2, 1979

problem (1.1)–(1.3) falls within the scope of [1]; see also [2, theorem 2]. An existence result for nonmonotone f, subject to severe growth restrictions, was announced (without proof) by Wong [11].

In the present work we are able to handle arbitrary continuous functions $f(x,u)$ which have the same sign as u. Our approach, carried out in Sections 2–4, involves some ideas previously used by Strauss [10] in the study of a class of hyperbolic equations related to (1.1).

2. The main result

The following basic assumptions are in force throughout the paper:

$$a \in W^{2,1}_{\text{loc}}([0,\infty); R), \tag{2.1}$$

$$a(0) > 0, \tag{2.2}$$

$$f \text{ is a real-valued continuous function on } \bar{\Omega} \times R, \tag{2.3}$$

$$uf(x,u) \geqq 0, \qquad \text{for all } x \in \Omega \text{ and } u \in R, \tag{2.4}$$

$$h \in W^{1,1}_{\text{loc}}([0,\infty); L^2(\Omega)), \tag{2.5}$$

$$u_0 \in H^1_0(\Omega), \tag{2.6}$$

$$g(.\,,u_0(.)) \in L^1(\Omega) \qquad \left(g(x,u) = \int_0^u f(x,s)ds, \qquad x \in \Omega, \quad u \in R \right). \tag{2.7}$$

Recall (see e.g. [3, p. 19]) that $W^{k,p}_{\text{loc}}([0,\infty); X)$ (X — a Banach space) consists of all locally absolutely continuous functions w from $[0,\infty)$ to X, whose derivatives $d^j w/dt^j$ (defined almost everywhere) are locally absolutely continuous for $j = 1, 2, \cdots, k-1$ and lie in $L^p_{\text{loc}}([0,\infty); X)$ for $j = 1, 2, \cdots, k$.

Our main result is

THEOREM 2.1. *Let the hypotheses* (2.1)–(2.7) *be satisfied. Then there exists a function* $u(t,x)$ *satisfying* (1.1) (*in the sense of distributions*), (1.2), (1.3) *and*

$$u \in L^\infty_{\text{loc}}([0,\infty); H^1_0(\Omega)) \cap C([0,\infty); L^2(\Omega)), \tag{2.8}$$

$$u_t \in L^\infty_{\text{loc}}([0,\infty); L^2(\Omega)) \cap L^1_{\text{loc}}([0,\infty); H^{-1}(\Omega) + L^1(\Omega)), \tag{2.9}$$

$$u_{tt} \in L^1_{\text{loc}}([0,\infty); H^{-1}(\Omega) + L^1(\Omega)), \tag{2.10}$$

$$f(.\,,u(.\,,.)) \in L^1_{\text{loc}}([0,\infty); L^1(\Omega)), \tag{2.11}$$

$$u(.\,,.)f(.\,,u(.\,,.)) \in L^1_{\text{loc}}([0,\infty); L^1(\Omega)). \tag{2.12}$$

REMARK 2.1. (i) From (2.8)–(2.10) it follows that u and u_t are weakly continuous maps of $[0,\infty)$ into $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively.

(ii) Since $a = 1$ satisfies (2.1), (2.2), the theorem provides solutions to the semilinear hyperbolic equation

$$u_{tt}(t,x) - \Delta u(t,x) + f(x, u(t,x)) = h_t(t,x), \qquad t > 0, \quad x \in \Omega,$$

with the boundary condition

$$u(t,x) = 0 \qquad \text{on } [0,\infty) \times \Gamma$$

and the initial conditions

$$u(0,x) = u_0(x), \quad u_t(0,x) = h(0,x), \qquad x \in \Omega.$$

Compare [10, theorem 2].

3. The approximating equation

For brevity in notation, we set: $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, $V' = H^{-1}(\Omega)$. We have $V \subset H \subset V'$, with dense and continuous inclusions. Moreover

(3.1) The injection of V into H is compact.

The norms in V and H will be denoted by $\|.\|$ and $|.|$ respectively. If $v \in V$ and $v' \in V'$ we use (v', v) to indicate their scalar product in the duality between V' and V; this coincides with their inner product in H, whenever $v' \in H$.

Let $A = -\Delta$ and remark that A is a linear, positive and symmetric operator from V into V'. Therefore $A = \partial\varphi$ (∂ = subdifferential), with

$$\varphi(u) = \tfrac{1}{2}(Au, u) = \tfrac{1}{2}\|u\|^2, \qquad u \in V. \tag{3.2}$$

Define A_H in H by

$$A_H v = Av, \qquad v \in D(A_H) = \{v \in V; Av \in H\}.$$

Obviously,

$$A_H = \partial\varphi_H, \qquad \varphi_H(v) = \begin{cases} \varphi(v), & v \in V, \\ +\infty, & v \in H \setminus V. \end{cases}$$

As A_H is maximal monotone, the Yosida approximations A_λ of A_H can be defined for $\lambda > 0$ by

$$A_\lambda = \lambda^{-1}(I - J_\lambda), \tag{3.3}$$

where I designates the identity on H and $J_\lambda = (I + \lambda A_H)^{-1}$. It is well-known that J_λ is a monotone contraction in H, while A_λ is maximal monotone and Lipschitz with constant $1/\lambda$. Moreover,

$$A_\lambda = \partial\varphi_\lambda,$$

$$\varphi_\lambda(u) = \varphi_H(J_\lambda u) + \frac{\lambda}{2}|A_\lambda u|^2 = \inf_{v\in H}\left\{\frac{|u-v|^2}{2\lambda} + \varphi_H(v)\right\}, \qquad u \in H. \tag{3.4}$$

We refer the reader to [4] for background material on monotone operators in Hilbert spaces.

On the other hand, in view of (2.3), (2.4) and [10, lemma 2.2] there is an approximating sequence of continuous functions $f_k : \bar{\Omega} \times R \to R$, $k = 1, 2, \cdots$, satisfying

$$\begin{gathered}\text{For each } k \text{ there exists } c_k > 0 \text{ such that}\\ |f_k(x,u) - f_k(x,v)| \leqq c_k |u - v|, \qquad \forall x \in \Omega, \quad \forall u, v \in R,\end{gathered} \tag{3.5}$$

$$\begin{gathered}f_k \text{ tends to } f, \text{ as } k \to \infty \text{ uniformly on } \Omega \times B,\\ \text{for any bounded subset } B \text{ of } R,\end{gathered} \tag{3.6}$$

$$uf_k(x,u) \geqq 0, \qquad \forall k, \quad \forall (x,u) \in \Omega \times R. \tag{3.7}$$

From (3.5) it follows that the operator $F_k : H \to H$, given by

$$(F_k u)(x) = f_k(x, u(x)), \qquad u \in H, \quad x \in \Omega,$$

is everywhere defined and Lipschitz continuous.

We now replace (1.1)–(1.3) by the following approximating problem in H:

$$\begin{aligned}&\text{(i)} \quad u_k'(t) + \textstyle\int_0^t a(t-s)(A_k + F_k)u_k(s)ds = h(t), \ t > 0 \ ('= d/dt),\\ &\text{(ii)} \quad u_k(0) = u_{k0}.\end{aligned} \tag{3.8}$$

Here A_k stands for the Yosida approximation of A_H corresponding to $\lambda = 1/k$ (see (3.3)) and u_{k0} is the sequence of truncated functions associated to u_0 in the usual way, i.e.,

$$u_{k0}(x) = \begin{cases} u_0(x), & \text{if } |u_0(x)| \leqq k,\\ k, & \text{if } u_0(x) > k,\\ -k, & \text{if } u_0(x) < -k.\end{cases}$$

Note (see [9]) that $u_{k0} \in V$ and $u_{k0} \to u_0$ ($k \to \infty$) in V, as well as a.e.

Inasmuch as $A_k + F_k$ is Lipschitzian on H, (3.8) has a unique solution $u_k \in W^{2,2}_{\mathrm{loc}}([0,\infty); H)$. We aim at establishing bounds on u_k, as $k \to \infty$.

First define $G_k : H \to R$ by

$$G_k(u) = \int_\Omega g_k(x, u(x))dx, \qquad u \in H;$$

$$g_k(x, v) = \int_0^v f_k(x, s)ds, \qquad (x, v) \in \Omega \times R.$$

Clearly (cf. (3.7)), G_k is nonnegative on H. In addition, one can readily show that $u \to G_k u$ is Fréchet differentiable on H and admits $F_k u$ as Fréchet differential at every point, i.e. for all $u \in H$,

$$G_k(u + h) - G_k(u) = (F_k(u), h) + o(|h|), \qquad \text{as } |h| \to 0.$$

Consequently,

$$(u_k'(t), F_k(u_k(t))) = \frac{d}{dt} G_k(u_k(t)), \qquad \text{a.e. on } (0,\infty). \tag{3.9}$$

We now form the (,) product of (3.8)(i) with $v_k(t) = (A_k + F_k)u_k(t)$ and integrate over $(0, t)$, $0 \leqq t \leqq T$ (T arbitrary). Using (3.4), (3.8)(ii), (3.9), and [4, lemma 3.3] we obtain

$$\begin{aligned} \varphi_k(u_k(t)) - \varphi_k(u_{k0}) + G_k(u_k(t)) - G_k(u_{k0}) + \int_0^t (a * v_k(s), v_k(s))ds \\ = \int_0^t (h(s), v_k(s))ds, \qquad 0 \leqq t \leqq T, \end{aligned} \tag{3.10}$$

where $a * v(t) = \int_0^t a(t-s)v(s)ds$ and φ_k denotes φ_λ with $\lambda = 1/k$, cf. (3.4).

Since $\varphi_k \geqq 0$, $G_k \geqq 0$ and $\varphi_k(u) \leqq \varphi(u)$, $\forall u \in H$, (3.10) leads to

$$\begin{aligned} \int_0^t (a * v_k(s), v_k(s))ds \leqq \varphi(u_{k0}) + G_k(u_{k0}) + \left(h(t), \int_0^t v_k(s)ds\right) \\ - \int_0^t \left(h'(s), \int_0^s v_k(\tau)d\tau\right)ds. \end{aligned} \tag{3.11}$$

Calling on the properties of f, f_k, u_0, u_{k0}, it is easy to deduce (see [10]) that

$$G_k u_{k0} \to G u_0 = \int_\Omega g(x, u_0(x))dx \qquad (k \to \infty). \tag{3.12}$$

It should be noted that (3.12) actually holds for an appropriate subsequence G_{j_k} of G_k ($j_k \to \infty$), but we do not bother changing the notation. Next observe that (3.2) and the convergence of u_{k0} to u_0 in V give

$$\varphi(u_{k0}) \to \varphi(u_0). \tag{3.13}$$

Thus, employing (3.12), (3.13) and (2.5) in (3.11) yields

$$\int_0^t (a * v_k(s), v_k(s))ds \leqq C\left(1 + \max_{0 \leqq s \leqq t}\left|\int_0^s v_k(\tau)d\tau\right|\right), \qquad t \in [0, T]. \tag{3.14}$$

Here and in the sequel the letter C designates a finite positive constant, independent of k.

Now, let us mention that (2.1), (2.2) are stronger than Conditions (a_1) in [5, proposition (a)]. Hence a satisfies Conditions (a) of [5]. This information in conjunction with (3.14) implies

$$\left|\int_0^t v_k(s)ds\right| \leqq C, \qquad 0 \leqq t \leqq T \tag{3.15}$$

and

$$\left|\int_0^t (a * v_k(s), v_k(s))ds\right| \leqq C, \qquad 0 \leqq t \leqq T. \tag{3.16}$$

Since $a * v_k(t) = a(0)\int_0^t v_k(s)ds + \int_0^t a'(t-s)\int_0^s v_k(\tau)d\tau ds$, from (3.8)(i) and (3.15) it follows that

$$\{u_k'\} \text{ is bounded in } L^\infty(0, T; H). \tag{3.17}$$

Then combining (3.10), (3.12), (3.13), (3.15) and (3.16) we get

$$\varphi_k(u_k(t)) \leqq C, \qquad 0 \leqq t \leqq T. \tag{3.18}$$

By (3.4), this implies

$$\varphi_H(J_k u_k(t)) \leqq C, \quad 0 \leqq t \leqq T, \quad J_k = \left(I + \frac{1}{k}A_H\right)^{-1}$$

and consequently (cf. (3.2))

$$\{J_k u_k\} \text{ is bounded in } L^\infty(0, T; V). \tag{3.19}$$

Taking note of the fact that

$$|J_k u_k(t+h) - J_k u_k(t)| \leqq |u_k(t+h) - u_k(t)|,$$

we infer from (3.17) that

$$\{J'_k u_k\} \text{ is bounded in } L^\infty(0, T; H). \tag{3.20}$$

4. Proof of Theorem 2.1

The proof rests on the estimates of the preceding section. By (3.1), (3.19), (3.20) and the Ascoli theorem we may choose an infinite subsequence of the integers k (which we denote without loss of generality as the original sequence) such that

$$J_k u_k \to u \quad (k \to \infty), \qquad \text{strongly in } C([0, T]; H) \tag{4.1}$$

and

$$J_k u_k \to u, \qquad \text{weakly-star in } L^\infty(0, T; V). \tag{4.2}$$

Since $A_k u_k(t) = AJ_k u_k(t)$, (4.2) leads to

$$A_k u_k \to Au, \qquad \text{weakly-star in } L^\infty(0, T; V'). \tag{4.3}$$

Recall now that

$$|u_k(t) - J_k u_k(t)|^2 = \frac{1}{k^2}|A_k u_k(t)|^2.$$

This, (3.4), (3.18) and (4.1) give us

$$u_k \to u, \qquad \text{strongly in } C([0, T]; H). \tag{4.4}$$

From (3.17) and (4.4) it follows that for a subsequence of the original sequence (which we denote once more for simplicity of notation as $\{u_k\}$),

$$u'_k \to u_t = u', \qquad \text{weakly-star in } L^\infty(0, T; H). \tag{4.5}$$

It remains to establish the convergence of $F_k u_k(t)$. This is carried out by means of [10, theorem 1.1]. We view $\{F_k u_k(t)\}$ as a sequence in $L^1(0, T; L^1(\Omega)) = L^1(Q)$, $Q =]0, T[\times \Omega$. Firstly, by (4.4), (2.3) and (3.6) one has (for some subsequence again denoted as $\{u_k\}$)

$$\begin{aligned} &\text{(i)} \quad u_k(t, x) \to u(t, x), \qquad \text{a.e. } (t, x) \in Q, \\ &\text{(ii)} \quad f_k(x, u_k(t, x)) \to f(x, u(t, x)), \qquad \text{a.e. on } Q. \end{aligned} \tag{4.6}$$

Then we will prove that

$$\int_0^T \int_\Omega |u_k(t, x)| \; |f_k(x, u_k(t, x))| \, dxdt \leqq C < \infty,$$

or, equivalently (see (3.7))

$$\int_0^T \int_\Omega u_k(t,x) f_k(x, u_k(t,x))\,dx\,dt \leqq C. \tag{4.7}$$

We proceed by differentiating (3.8)(i) in H; this gives

$$u_k''(t) + a(0)v_k(t) + a' * v_k(t) = h'(t), \qquad \text{a.e. } t > 0. \tag{4.8}$$

Multiply (4.8) by $u_k(t)$ and integrate over $(0, T)$ to obtain

$$\begin{aligned}(u_k(T), u_k'(T)) - (u_{k0}, h(0)) - \int_0^T |u_k'(t)|^2 dt + a(0)\int_0^T (A_k u_k(t), u_k(t))dt \\ + a(0)\int_0^T (F_k u_k(t), u_k(t))dt + \int_0^T (a' * v_k(t), u_k(t))dt \\ = \int_0^T (h'(t), u_k(t))dt.\end{aligned} \tag{4.9}$$

Inasmuch as $u_k(t)$ and $u_k'(t)$ are uniformly bounded on $[0, T]$ (cf. (3.17), (4.4)) and $(A_k u_k(t), u_k(t)) \geqq 0$, from (4.9) it follows that

$$a(0)\int_0^T (F_k u_k(t), u_k(t))dt \leqq C\left(1 + \int_0^T |a' * v_k(t)|\,dt\right). \tag{4.10}$$

Taking into account (2.1) we may write

$$a' * v_k(t) = a'(0)\int_0^t v_k(s)ds + \int_0^t a''(t-s)\int_0^s v_k(\tau)d\tau ds.$$

In view of (3.15), this yields $|a' * v_k(t)| \leqq C$, $0 \leqq t \leqq T$ and consequently (cf. (4.10), (2.2)), (4.7) holds.

Now, by virtue of (3.6), (4.6)(ii) and (4.7), theorem 1.1 in [10] enables us to deduce that

$$f(x, u(t,x)) \in L^1(Q) \tag{4.11}$$

and

$$F_k u_k(t) \to f(.\,, u(t,.)) = Fu(t), \qquad \text{strongly in } L^1(0, T; L^1(\Omega)). \tag{4.12}$$

We can finally show that $u(t)$ satisfies (1.1)–(1.3) as well as (2.8)–(2.12). Recall first that T was arbitrarily chosen. By (4.3)–(4.5) and (4.12), passage to the limit in (3.8) yields (1.1) and (1.3). Then (4.2) and (4.4) imply (1.2) and (2.8). Assertions (2.9)–(2.11) are straightforward consequences of (4.3), (4.5), (4.11) and (2.5). As regards (2.12), it follows easily from (3.7), (4.6) and (4.7) via Fatou's lemma. The proof is complete.

5. Heat conduction in materials with memory

In this section we suggest a special heat flow model to which our previous theory applies. We let Ω be the region of three-dimensional Euclidean space occupied by a rigid, stationary heat conductor. Let $u(x,t)$, $e(x,t)$, $q(x,t)$ and $r(x,t)$ $(x \in \Omega,\ -\infty < t < \infty)$ denote the absolute temperature, internal energy, heat flux and heat supply respectively. The energy balance equation assumes the form

$$e_t = -\nabla q + r. \tag{5.1}$$

Gurtin and Pipkin [6] have developed a general theory of heat conduction for nonlinear materials with memory. When this theory is linearized, it yields the constitutive equations

$$e(x,t) = cu(x,t) + \int_0^\infty \alpha(s)u(x,t-s)ds, \tag{5.2}$$

$$q(x,t) = -\int_0^\infty \beta(s)\nabla u(x,t-s)ds, \tag{5.3}$$

for isotropic materials. Here $c > 0$ denotes the instantaneous heat capacity, while α and β stand for the respective energy and heat flux relaxation functions.

We can treat a partially nonlinear variant of (5.2) and (5.3). We keep (5.3) and replace (5.2) by

$$e(x,t) = cu(x,t) + \int_0^\infty \alpha(s)f(x,u(x,t-s))ds \qquad (f:\Omega\times R \to R). \tag{5.4}$$

(Mac Camy [7] considered the one-dimensional version of (5.2), (5.3) with a nonlinearity in the heat flux relation.)

Without loss of generality we may suppose the material is at zero temperature and internal energy up to time $t = 0$. (Nonzero initial histories can be incorporated into the forcing term.) Then (5.1), (5.3) and (5.4) lead to

$$cu_t(x,t) + \alpha(0)f(x,u(x,t)) + \int_0^t \alpha'(t-s)f(x,u(x,s))ds$$
$$= \int_0^t \beta(t-s)\Delta u(x,s)ds + r(x,t), \qquad t>0, \quad x\in\Omega.$$

If we take $\alpha(0) = 0$ (this is reasonable from the physical viewpoint, see [8]) and $\alpha'(t) = k\beta(t)$, $k > 0$ we arrive at an equation of the form (1.1) where

$$a(t) = \frac{1}{c}(1+k)\beta(t) \quad \text{and} \quad h = \frac{1}{c}r.$$

Consequently, one can apply Theorem 2.1, provided that (2.1)–(2.7) hold. Note that in this context, conditions (2.1), (2.2) seem quite natural, cf. [8].

6. Final remarks

REMARK 6.1. The procedure discussed in Sections 3 and 4 works equally well when

(i) $-\Delta$ is replaced by a linear positive, symmetric, strongly elliptic operator $L : H_0^m(\Omega) \to H^{-m}(\Omega)$;

(ii) Condition (2.4) is weakened to

$$f(x, u) \geqq f_0(x), \qquad \forall x \in \Omega \quad \text{and} \quad u \geqq M,$$

and correspondingly

$$f(x, u) \leqq f_0(x), \qquad \forall x \in \Omega \quad \text{and} \quad u \leqq -M,$$

for some $M \geqq 0$ and $f_0 \in C(\bar{\Omega})$.

REMARK 6.2. By a well-known device (see e.g. [7]), Eq. (1.1) can be reduced to

$$\frac{1}{a(0)} u_{tt} - \Delta u + f(x, u) + r(0) u_t + r' * u_t = H(t, x), \tag{6.1}$$

where r is the resolvent of a', that is, the solution of

$$a(0) r(t) + \int_0^t a'(t-s) r(s) ds = -\frac{a'(t)}{a(0)},$$

and H depends on h and r.

The left-hand side of (6.1) is seen to be a linear perturbation of the operator considered by Strauss [10]. Consequently, starting from (6.1) one could also develop an existence theory, along the lines of [10].

REMARK 6.3. Let u be the solution of (1.1)–(1.3) constructed in the proof of Theorem 2.1. Results concerning boundedness and asymptotic behaviour of u on $(0 \leqq t < \infty)$ can easily be derived at the expense of stronger assumptions on a and h. For example, replace (2.1), (2.2) by

$$\begin{gathered} a \in C^2(0, \infty) \cap C^1[0, \infty); \\ a(0) > 0; \quad (-1)^k a^k(t) \geqq 0, \qquad k = 0, 1, 2, \quad t > 0. \end{gathered} \tag{6.2}$$

Suppose also that h satisfies (2.5) and ($| \ . \ |$ stands for the norm in $H = L^2(\Omega)$)

(6.3) $$|h(t)| \leqq \delta a(t), \quad |h'(t)| \leqq -\delta a'(t), \qquad t > 0,$$

for some $\delta > 0$.

In this case, the technique of [5, p. 718] (see especially the remarks at the end of the proof of theorem 3 in [5]) enables us to deduce

THEOREM 6.1. *Assume that* (2.3)–(2.7), (6.2) *and* (6.3) *hold. Then problem* (1.1)–(1.3) *has a solution u satisfying* (*besides* (2.8)–(2.12))

$$u \in L^\infty(0, \infty; V),$$

$$\sup_{0 \leqq t < \infty} Q_a(v; t) < \infty,$$

$$|a * v(t)|^2 \leqq \liminf_{k \to \infty} 2a(0) Q_a(v_k; t), \quad t \geqq 0,$$

where

$$v_k(t) = (A_k + F_k)u_k(t), \qquad v(t) = Au(t) + Fu(t),$$

while

$$Q_a(v; t) = \frac{a(t)}{2} \left| \int_0^t v(s)ds \right|^2 - \frac{1}{2} \int_0^t a'(\tau) \left| \int_0^\tau v(s)ds \right|^2 d\tau$$

$$-\frac{1}{2} \int_0^t a'(t-\tau) \left| \int_\tau^t v(s)ds \right|^2 d\tau + \frac{1}{2} \int_0^t \int_0^\tau a''(\tau - s) \left| \int_s^\tau v(\sigma)d\sigma \right|^2 ds d\tau.$$

(*Similarly one defines* $Q_a(v_k; t)$.)

Note that since $\int_0^t v(s)ds \in H$, $t \geqq 0$, the above expressions make sense.

Suppose now (cf. [7]) that

(6.4) $$a \in C^2[0, \infty); \qquad a(0) > 0; \qquad a'(0) < 0,$$

(6.5) $$t^j a^{(k)}(t) \in L^1(0, \infty), \qquad k = 0, 1, 2; \quad j \leqq 3 + N \text{ for some } N \geqq 0,$$

(6.6) $$\operatorname{Re} \hat{a}(i\eta) > 0 \quad \text{for all } \eta.$$

Here $\hat{a}$ denotes the Laplace transform of a. (From (6.5) it follows that $\hat{a}(s)$ is of class $C^{(N+3)}$ in $\operatorname{Re} s \geqq 0$.)

Let us strengthen (2.5) to

(6.7) $$h \in W^{1,2}([0, \infty); L^2(\Omega)).$$

If we now use Eq. (3.8), written under the form (6.1), and follow the procedure of Mac Camy [7, lemma 4.1], we arrive at

THEOREM 6.2. *Let the conditions* (2.3), (2.4), (2.6), (2.7) *and* (6.4)–(6.7) *be satisfied. Let u be the solution of* (1.1)–(1.3) *given by Theorem* 2.1. *Then*

$$\lim_{t\to\infty}\int_\Omega u^2(t,x)dx = 0.$$

ACKNOWLEDGEMENT

The author is indebted to the referee for his valuable suggestions.

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